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Boundary value problem problem prescribed in Minkowski space

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Abstract. We are interested in a second-order nonlinear problem under homogeneous dirichlet boundary conditions. By combining the method of lower- and upper-solutions, the variational method and the theory of topological degree, we obtain results on the existence of solutions and the multiplicity of solutions when the problem admits well-ordered or unordered lower- and upper-solutions, or a single lower- or upper-solution.

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1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} -\operatorname{div}(\Phi(\nabla z)) = f(x, z, \nabla z) \quad \text{in } \Gamma \\ z = 0 \quad \text{on } \partial\Gamma, \end{cases} \tag{1.1}$$

where, $\Phi : B_{\mathbb{R}^N}(1) \to \mathbb{R}^N$, with $B_{\mathbb{R}^N}(1)$ an open ball of $\mathbb{R}^N$ centred at 0 with raduis 1, is a homeomorphism which is the derivative of a concave function and defined by $\Phi(\nabla u) = a(\nabla u)\dfrac{\nabla u}{\sqrt{1 - \|\nabla u\|^2}}$, where $a : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, $f : \Gamma \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies Caratheodory conditions.

The scope of this study is relativity. For example, the dynamics of a charged particle in an electric and magnetic field when the velocities of the particles are relativistic. In addition, this problem can be a model of certain phenomena in classical physics, such as the shape of a film of soap stretched on a string, capillarity and wetting phenomena, the curved surfaces of liquids in contact with a solid, the growth of crystals, plasma physics, non-linear optics and so on. Over the last two decades, several authors have taken an interest in this type of problem. See, for example, the articles [1, 2, 3, 4, 5, 6, 7, 8] and references contained therein. In [1], the authors are interested in positive solutions defined on all $\mathbb{R}^N$ and vanishing at infinity whereas in (1) as in other references, the authors are interested in positive solutions defined on a bounded part of $\mathbb{R}^N$. Contrary to (1.1) and [2, 6, 7], [3, 4, 5, 8] deal with positive radial solutions. These papers deal with the prescribed mean curvature problem in a Minkowski space governed by the homogeneous differential operator $-\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-\|\nabla u\|^2}}\right)$ while, in this paper, we consider the non-homogeneous differential operator $-\operatorname{div}\left(a(\nabla u)\dfrac{\nabla u}{\sqrt{1-\|\nabla u\|^2}}\right)$. Consequently, this work generalises previous work. For such a problem, using the same approach as [7], we will establish the existence and multiplicity of solutions. In fact, by combining the method of lower and upper solutions, variational method and theory of topological degree, we obtain the existence of a solution when the problem admits a unique lower solution or a unique upper solution; we obtain a multiplicity of solutions when the problem admits a lower solution and upper solutions which may or not be well-ordered.

The rest of the article is organised as follows: in the second section, we establish three auxiliary results, the first two using the method of lower and upper solutions combined with the Leray-Schauder topological degree theory, and the third using the method of lower and upper solutions combined with the variational method. The third section is devoted to our main results. In the section four, we give some application examples. The last section is reserved for the conclusion.

2. Auxiliary results

Our assumptions about the problem data are as follows:

(H_Γ): Γ is a bounded subset of $\mathbb{R}^N$ which has a boundary $\partial\Gamma$ of class C^2,

(H_Φ) : $\Phi : B_{\mathbb{R}^N}(1) \to \mathbb{R}^N$ is a monotone homeomorphism and gradient of a concave function φ such that $\Phi(0) = 0$.

Remark 2.1. Some functions of form $\Phi(u) = a(u)\dfrac{u}{\sqrt{1-\|u\|^2}}$, where $a : \mathbb{R}^N \to \mathbb{R}$, satisfy hypothesis ($H_\Phi$). For example, the hypothesis (H_Φ) is satisfied when a is defined by :

$$a(u) = -\frac{4+u^2+u^4}{(1+u^2)^2} \text{ or } a(u) = -\frac{4+2u^2+3u^4+u^6}{(1+u^2)\sqrt{1+(1+u^2)^2}}.$$

In each of theses cases, $\Phi = \nabla\varphi$, with φ defined on $\mathbb{R}^N$ by $\varphi(u) = \left(1+\frac{1}{1+u^2}\right)\sqrt{1-u^2}$ and $\varphi(u) = \dfrac{\sqrt{1+(1+u^2)^2}}{1+u^2}\sqrt{1-u^2}$ respectively. We can check that φ is concave and Φ is a monotone homeomorphism such that $\Phi(0) = 0$.

(H_f) : $f : \Gamma \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:

(i): for all $(r,s) \in \mathbb{R}^2$, $x \mapsto f(x,r,s)$ is measurable;

(ii): for almost all $x \in \Gamma$, $(r,s) \mapsto f(x,r,s)$ is continuous;

(iii): for all $\gamma > 0$, there exist $g_\gamma \in L^\infty(\Gamma)^+$ such that for almost all $x \in \Gamma$ and all $(r,s) \in \mathbb{R}^2$, with $|r|,|s| < \gamma$, $|f(x,r,s)| < g_\gamma(x)$.

Remark 2.2. f is said to be a $L^\infty-$ Caratheodory function when it verifies hypotheses (i),(ii) and (iii).

Let's define our notion of solution of the problem (1.1).

Definition 2.3. We call the *solution* of the problem(1.1) any function z of $C^{0,1}(\overline{\Gamma})$ such that

- $\Phi(\nabla z) \in W^{1,1}(\Gamma)$,
- $\|\nabla z\|_\infty < 1$,
- $\int_\Gamma \Phi(\nabla z(x)).\nabla w(x)dx = \int_\Gamma f(x,z(x),\nabla z(x))w(x)dx$, for all $w \in W_0^{1,1}(\Gamma)$.

Remark 2.4. $\|z\|_\infty < \frac{1}{2}\text{diam}\Gamma$ (See remark 1 of [7]).

Our definition of lower solution and upper solution of the problem (1.1) are the following:

Definition 2.5. We call the *lower solution* of the problem(1.1) any function σ of $C^{0,1}(\overline{\Gamma})$

- $\Phi(\nabla \sigma) \in W^{1,1}(\Gamma)$,
- $\|\nabla \sigma\|_\infty < 1$,
- $\int_\Gamma \Phi(\nabla \sigma(x)).\nabla w(x)dx \leq \int_\Gamma f(x,\sigma(x),\nabla \sigma(x))w(x)dx$, for all $w \in W_0^{1,1}(\Gamma)$ such that $w \geq 0$.

Definition 2.6. We call the *upper solution* of the problem(1.1) any function ρ of $C^{0,1}(\overline{\Gamma})$ such that

- $\Phi(\nabla \rho) \in W^{1,1}(\Gamma)$,
- $\|\nabla \rho\|_\infty < 1$,
- $\int_\Gamma \Phi(\nabla \rho(x)).\nabla w(x)dx \geq \int_\Gamma f(x,\rho(x),.\nabla \rho(x))w(x)dx$, for all $w \in W_0^{1,1}(\Gamma)$ such that $w \geq 0$.

Remark 2.7. If we set $v = f(.,u,\nabla u) \in L^\infty(\Gamma)$, then the following problem :

$$\begin{cases} -\operatorname{div}(\Phi(\nabla z)) = v \text{ in } \Gamma \\ z = 0 \text{ on } \partial\Gamma. \end{cases} \tag{2.1}$$

has a unique solution $u \in W^{2,p}(\Gamma), p \geq 1$ (See lemma 2.2 of [6]). Let's set

$$D = \{w \in C^{0,1}(\overline{\Gamma}) : \|\nabla w\|_\infty \leq 1;\ w = 0 \text{ on } \partial\Gamma\} \tag{2.2}$$

and define the following functional:

$$\phi_v : w \mapsto \int_\Gamma \varphi(\nabla w(x))dx + \int_\Gamma v(x)w(x)dx \tag{2.3}$$

for all $w \in D$. For any $y \in D$, multiplying (2.1) by $z-y$ and integrating by parts, we obtain:

$$\int_\Gamma \Phi(\nabla z(x)).\nabla(z-y)(x)dx = \int_\Gamma v(x)(z-y)(x)dx \tag{2.4}$$

Let's consider the concavity function $y \mapsto \varphi(y)$ such that $\nabla\varphi = \Phi$. We have

$$\int_\Gamma \varphi(\nabla y(x))dx - \int_\Gamma \varphi(\nabla z(x))dx \leq \int_\Gamma \Phi(\nabla z(x)).\nabla(z-y)(x)dx \tag{2.5}$$

From (2.4) and (2.5), it follows:

$$\varphi_v(y) \leq \varphi_v(z)$$

This means that both u and z are variational solutions of (2.1) in the sense of [9]. Then Lemma 1.2 of [1] implies that $u = z$. Thus $u \in W^{2,p}(\Gamma)$) for all finite $p \geq 1$.

Let's define the operator $T : C^1(\overline{\Gamma}) \to C_0^1(\overline{\Gamma})$ which sends any function $v \in C^1(\overline{\Gamma})$ onto the unique solution $u \in W^{2,p}(\Gamma)$, for all finite $p \geq 1$, of the problem

$$\begin{cases} -\operatorname{div}(\Phi(\nabla z)) = v \text{ in } \Gamma \\ z = 0 \text{ on } \partial\Gamma. \end{cases} \tag{2.6}$$

If u is a fixed point of T, then u is a solution of (1.1). Let us consider the following open ball :

$$B = \left\{ y \in C_0^1(\overline{\Gamma}) : \|\nabla y\|_\infty < 1 \right\}.$$

Lemma 2.8. *Suppose that (H_f) and (H_Φ) hold. Then the operator T is continuous, completely continuous and deg$(I-T, B, 0) = 1$, where I is the identity operator.*

Proof. The proof is similar to the one of Lemma 3.1 of [7]. □

Proposition 2.9. *Suppose that (H_f) and (H_Φ) hold. If the problem (1.1) admits a well ordered lower and upper solution σ and ρ, then it has solutions z_1, z_2 such that $\sigma \leq z_1 \leq z_2 \leq \rho$ and $z_1 \leq z \leq z_2$ for any solution z of (1.1) in the functional interval $[\sigma, \rho]$. Moreover, if σ and ρ are strict, then*

$$deg(I - T, \Lambda, 0) = 1, \tag{2.7}$$

where

$$\Lambda = \left\{ z \in C_0^1(\overline{\Gamma}) : \sigma \ll z \ll \rho \quad \text{and} \quad \|\nabla z\|_\infty < 1 \right\}$$

Proof. proof will be established in several parts.

Claim 1. *The problem (1.1) has a solution z belonging to the funtional interval $[\sigma, \rho]$.*

Proof. Consider the following truncated function for a.e $x \in \Gamma$ and all $u \in \mathbb{R}$:

$$\overline{f}(x,u,v) = \begin{cases} f(x, \sigma(x), \nabla\sigma(x)) & \text{if } u \leq \sigma(x) \\ f(x,u,v) & \text{if } \sigma(x) < u < \rho(x) \\ f(x, \rho(x), \nabla\rho(x)) & \text{if } u \geq \rho(x). \end{cases}$$

Now, consider the following modified problem:

$$\begin{cases} -\ \operatorname{div}\ \Phi(\nabla z(x)) = \overline{f}(x, z(x), \nabla z(x)) \quad \text{in } \Gamma \\ z = 0 \text{ on } \partial\Gamma. \end{cases} \tag{2.8}$$

Consider the function $(\sigma - z)^+$ define on Γ by

$$(\sigma - z)^+(x) = \begin{cases} (\sigma - z)(x) & \text{if } \sigma(x) > z(x) \\ 0 & \text{if } \sigma(x) \leq z(x). \end{cases}$$

We also define the function $\nabla(\sigma - z)^+$ as follow:

$$\nabla(\sigma - z)^+(x) = \begin{cases} \nabla(\sigma - z)(x) & \text{if } \sigma(x) > z(x) \\ 0 & \text{if } \sigma(x) \leq z(x). \end{cases}$$

Recall the following definition of lower solution of (1.1).

$$\int_\Gamma \Phi(\nabla\sigma(x)).\nabla w(x)dx \leq \int_\Gamma f(x,\sigma(x),\nabla\sigma(x))w(x)dx \tag{2.9}$$

Multiply (2.8) by $(\sigma - z)^+$ and integrate by parts on Γ. We obtain:

$$\int_0^b \Phi(\nabla z(x)).[\nabla(\sigma - z)]^+(x)dx = \int_0^b \overline{f}(x, z(x), \nabla z(x)). \tag{2.10}$$

Replace w by $(\sigma - z)^+$ and subtracting (2.9) and (2.8) we obtain:

$$\begin{aligned} &\int_0^b \left(\Phi(\nabla\sigma(x)) - \Phi(\nabla z(x))\right).[\nabla(\sigma - z)]^+(x)dx \\ &\leq \int_0^b \left(\overline{f}(x,\sigma(x),\nabla\sigma(x)) - f(x,z(x),\nabla z(x))\right)(\sigma - z)^+(x)dx. \end{aligned} \tag{2.11}$$

Monotony hypothesis on Φ implies that

(2.12)

$$\begin{aligned} &\int_0^b \left(\Phi(\nabla\sigma(x)) - \Phi(\nabla z(x))\right).[\nabla(\sigma - z)]^+(x)dx \\ &+ (\Phi(\nabla\sigma(0)) - \Phi(\nabla z(0))(\sigma - z)^+(0) - (\Phi(\nabla\sigma(b)) - \Phi(\nabla z(b)))(\sigma - z)^+(b) \geq 0. \end{aligned}$$

We deduce that

$$\begin{aligned} &\int_\Lambda \left(\Phi(\nabla\sigma(x)) - \Phi(\nabla z(x))\right).[\nabla(\sigma - z)]^+(x)dx \\ &+ (\Phi(\nabla\sigma(0)) - \Phi(\nabla z(0))(\sigma - z)^+(0) - (\Phi(\nabla\sigma(b)) - \Phi(\nabla z(b)))(\sigma - z)^+(b) \geq 0, \end{aligned}$$

where $\Lambda = \{x \in \Gamma : \sigma(x) \geq z(x)\}$.

Furthermore, on Λ, $\overline{f}(x,\sigma(x),\nabla\sigma(x)) - f(x,z(x),\nabla z(x)) = 0$. Then (2.10) implies

(2.13)

$$\begin{aligned} &\int_\Lambda \left(\Phi(\nabla\sigma(x)) - \Phi(\nabla z(x))\right).[\nabla(\sigma - z)]^+(x)dx \\ &+ (\Phi(\nabla\sigma(0)) - \Phi(\nabla z(0))(\sigma - z)^+(0) - (\Phi(\nabla\sigma(b)) - \Phi(\nabla z(b)))(\sigma - z)^+(b) = 0. \end{aligned}$$

The strict monotonicity of Φ and Φ_p imply $(\sigma - z)^+ = 0$. Thus $z \geq \sigma$. Similarly, we show that $z \leq \rho$. □

Claim 2. *Problem* (1.1) *has a solution in* $[\sigma, \rho]$.

Proof. Let us consider the operator $T^* : C^1(\overline{\Gamma}) \to C_0^1(\overline{\Gamma})$ which sends any function of $v \in C^1(\Gamma)$ onto the unique solution z of

$$\begin{cases} -\left(\Phi(\nabla z(x))\right)' = \overline{f}(x, v(x), v'(x)) & \text{in } \Gamma \\ z = 0 \text{ on } \partial\Gamma. \end{cases} \tag{2.14}$$

By lemma 2.8,

$$\deg(I - T^*, B, 0) = 1. \tag{2.15}$$

Then T^* has a fixed point z which is solution of (2.14). By the above arguments $z \in [\sigma, \rho]$ and z is solution of (1.1).

□

Claim 3. *The problem* (1.1) *has extremal solutions.*

Proof. Let us consider the set $\Omega = \left\{z \in C_0^1(\overline{\Gamma}) : z = Tz \text{ and } \sigma \leq z \leq \rho\right\}$. By Lemma 2.8, T is compact. The proof of claim 1 shows that Γ is nonempty. To show that the set Ω has a minimal element, consider the following set:

$$U_z = \{u \in \Omega : u \leq z\}.$$

Let $z_1, z_2 \in U_z$. Then, $\min\{z_1, z_2\} \geq \sigma$. By the proof of claim 1, there exists a solution z of (1.1) such that $\sigma \leq z \leq \min\{z_1, z_2\} \leq \rho$. Then $U_{z_1} \cap U_{z_2} \neq \varnothing$.The compactness of Ω implies that there exists $v \in \bigcap_{z \in \Omega} U_z$. Therefore, v is the minimal solution of (1.1) belonging to $[\sigma, \rho]$. By analogous reasoning, we show that (1.1) admits a maximal solution in $[\sigma, \rho]$. □

Claim 4. *deg*$(I - T, \Lambda, 0) = 1$.

Proof. Suppose that (1.1) admits a strict lower solution σ and a strict upper solution ρ. Then there exists a solution z of (1.1) which satisfies $\sigma < z < \rho$. As consequence, the open set Λ is nonempty and bounded subset of $C^1(\Gamma)$ such that there no fixed point of T and T^* on the boundary $\partial\Lambda$. In addition, T and T^* coincide in Λ. Whence

$$\deg(I - T^*, \Lambda, 0) = \deg(I - T, \Lambda, 0).$$

T^* has no fixed point in $\overline{B}_{R+M(b+1)} \setminus \Lambda$. Then excision property of the degree and (2.15) led to

$$\deg(I - T^*, \Lambda, 0) = \deg(I - T^*, B, 0).$$

Therefore

$$\deg(I - T, \Lambda, 0) = 1.$$

□

□

Proposition 2.10. *Suppose that* (H_f) *and* (H_Φ) *hold. If the problem* (1.1) *admits a strict lower solution* σ *and a strict upper solution* ρ, *with* $\sigma \nleq \rho$, *then it admits at least three solutions* z_1, z_2, z_3 *such that*

$$z_1 \leq z_2 \leq z_3, \quad z_1 \ll \rho, \quad z_1 \ngeq \sigma, \quad z_2 \nleq \rho, \quad z_3 \gg \sigma. \tag{2.16}$$

Proof. Let us consider the function f_λ defined as follows

$$f_\lambda(x,r,s)=\begin{cases} \text{ar+bs} & \text{if } \lambda<|r|<\lambda+1\\ f(x,r,s) & \text{if } |r|\le\lambda\\ 0 & \text{if } |r|\ge\lambda+1\end{cases}$$

where a,b are real constants, $\lambda=\max\left\{\|\sigma\|_\infty,\|\rho\|_\infty,\frac{1}{2}\text{diam}\Gamma\right\}$. f_λ is a L^p-Carathéodory function. Let us consider the following modified problem: :

$$\begin{cases} -\left(\Phi(\nabla z(x))\right)'=f_\lambda(x,z(x),\nabla z(x)) & \text{in } \Gamma\\ z=0 \text{ on } \partial\Gamma.\end{cases} \tag{2.17}$$

The choice of λ implies that any solution of (2.17) is solution of (1.1). σ and ρ are also strict lower solution and strict upper solution of (2.17) respectively. The constants $\overline{\sigma}=-\lambda-1$ and $\overline{\rho}=\lambda+1$ are also strict lower solution and strict upper solution of (2.17) respectively. We consider the following subsets of $C^1(\Gamma)$:

$$\Pi_{\overline{\sigma},\rho}=\left\{z\in C_0^1(\overline{\Gamma}):\overline{\sigma}\ll z\ll\rho \text{ and } \|\nabla z\|_\infty<1\right\},$$

$$\Pi_{\sigma,\overline{\rho}}=\left\{z\in C_0^1(\overline{\Gamma}):\sigma\ll z\ll\overline{\rho} \text{ and } \|\nabla z\|_\infty<1\right\},$$
$$\Pi_{\overline{\rho},\overline{\sigma}}=\left\{z\in C_0^1(\overline{\Gamma}):\overline{\sigma}\ll z\ll\overline{\rho} \text{ and } \|\nabla z\|_\infty<1\right\},$$

where $u\ll v$ means that there is $\varepsilon>0$ such that $u(x)+\varepsilon\text{dist}(x,\partial\Gamma)\le v(x)$ for every $x\in\Gamma$. We have $\Pi_{\overline{\sigma},\rho}\subset\Pi_{\overline{\rho},\overline{\sigma}}$ and $\Pi_{\sigma,\overline{\rho}}\subset\Pi_{\overline{\rho},\overline{\sigma}}$ and $\Pi_{\overline{\sigma},\rho}\cap\Pi_{\sigma,\overline{\rho}}=\varnothing$. As a result,

$$0\notin(I-T_\lambda)\left(\partial\Pi_{\overline{\sigma},\overline{\rho}}\cup\partial\Pi_{\sigma,\overline{\rho}}\cup\partial\Pi_{\overline{\rho},\overline{\sigma}}\right) \tag{2.18}$$

because σ and $\overline{\sigma}$ are strict lower solutions of (2.17) and $\overline{\rho}$ and ρ are strict upper solutions of (2.17). The operator $T_\lambda: C^1(\Gamma)\to C_0^1(\Gamma)$ sends any function $v\in C^1(\Gamma)$ on the unique solution $z\in C^1(\Gamma)$ of

$$\begin{cases} -\left(\Phi(\nabla z(x))\right)'=f_\lambda(x,v(x),v'(x)) & \text{in } \Gamma\\ z(0)=(z(b)=0.\end{cases} \tag{2.19}$$

Let us consider the open set

$$A=\Pi_{\overline{\sigma},\overline{\rho}}\setminus\overline{(\Pi_{\sigma,\overline{\rho}}\cup\Pi_{\overline{\sigma},\rho})}$$

(2.18) and the excision property of the degree led to

$$\deg\left(I-T_\lambda,\Pi_{\overline{\sigma},\overline{\rho}},0\right)=\deg\left(I-T_\lambda,\Pi_{\overline{\sigma},\overline{\rho}}\setminus\left(\partial\Pi_{\overline{\sigma},\rho}\cup\partial\Pi_{\sigma,\overline{\rho}}\right),0\right)$$

Then the additivity property of the degree yields

$$\deg\left(I-T_\lambda,\Pi_{\overline{\sigma},\overline{\rho}},0\right)=\deg\left(I-T_\lambda,\Pi_{\sigma,\overline{\rho}},0\right)+\deg\left(I-T_\lambda,\Pi_{\overline{\sigma},\rho},0\right)+\deg\left(I-T_\lambda,A,0\right)$$

By proposition 2.9, we have

$$\deg\left(I-T_\lambda,\Pi_{\overline{\sigma},\overline{\rho}},0\right)=\deg\left(I-T_\lambda,\Pi_{\sigma,\overline{\rho}},0\right)=\deg\left(I-T_\lambda,\Pi_{\overline{\sigma},\rho},0\right)=1.$$

Using the above arguments, we obtain

$$\deg\left(I-T_\lambda,A,0\right)=-1.$$

As result the operator T_λ admits three distinct fixed points z_1,z_2,z_3 such that

$$z_1\in\Pi_{\overline{\sigma},\rho}\qquad z_2\in\Pi_{\sigma,\overline{\rho}}\qquad z_3\in A.$$

Then

$$z_1 \ll \rho \qquad z_1 \not\geq \sigma, \qquad z_2 \not\leq \rho, \qquad z_3 \gg \sigma$$

We know that the problem (2.19) admits extremal solutions v, w in $[\overline{\sigma}, \overline{\rho}]$. Suppose that $v = z_1$ and $w = z_3$. Then (2.19) and (1.1) have three distincts solutions which satisfy (3.4). □

Let us define in D the following operator:

$$\psi(r) = \int_\Gamma \varphi(r'(x))dx + \int_\Gamma F(x, r(x), r'(x))dx \tag{2.20}$$

where $F(x, r, s) = \int_0^r f(x, r, s)dr$.

Proposition 2.11. *Suppose that (H_f) and (H_Φ) hold. Suppose there exist a lower solution σ and an upper solution ρ, with $\sigma \leq \rho$. Then there is a solution z that maximizes the functional Ψ over the set $\{z \in D : \sigma \leq z \leq \rho\}$, where D is defined by (2.2).*

Proof. Consider the following functional $\overline{\Psi}$ defined over the set D by

$$\overline{\Psi}(r) = \int_\Gamma \varphi(r'(x))dx + \int_\Gamma \overline{F}(x, r(x), r'(x))dx$$

where $\overline{F}(x, r, s) = \int_0^r \overline{f}(x, r, s)dr$. By the proof of the proposition 1 of [9], there is $z \in D$ which maximizes $\overline{\psi}$ over D. This means that

$$\int_\Gamma \varphi(\nabla z(x))dx + \int_\Gamma F(x, z(x), \nabla z(x))dx \geq \int_\Gamma \varphi(r'(x))dx + \int_\Gamma F(x, r(x), r'(x))dx \tag{2.21}$$

for all $r \in D$. Let us choose r such that $r = z + \kappa(j - z)$ where $(\kappa, j) \in [0, 1] \times D$. By replacing r with its expression in (2.20) and then using concavity, we obtain

$$\begin{aligned}
&\int_\Gamma \varphi(\nabla z(x))dx + \int_\Gamma F(x, z(x), \nabla z(x))dx \\
&\geq \int_\Gamma \varphi(\nabla z(x) + \kappa\nabla(j - z)(x))dx + \int_\Gamma F(x, z(x) + \kappa(j - z)(x), \nabla z(x) + \kappa\nabla(j - z)(x))dx \\
&\geq \kappa \int_\Gamma \varphi(\nabla j(x))dx + (1 - \kappa) \int_\Gamma \varphi(\nabla z(x))dx \\
&+ \int_\Gamma F(x, z(x) + \kappa(j - z)(x), \nabla z(x) + \kappa\nabla(j - z)(x))dx
\end{aligned} \tag{2.22}$$

From (2.22), we have

$$\begin{aligned}
&\int_\Gamma \varphi(\nabla z(x))dx - \int_\Gamma \varphi(\nabla j(x))dx \\
&\geq \int_\Gamma \frac{F(x, z(x) + \kappa(j - z)(x), \nabla z(x) + \kappa\nabla(j - z)(x)) - F(x, z(x), \nabla z(x))}{\kappa} dx \\
&\geq \int_\Gamma \frac{F(x, z(x) + \kappa(j - z)(x), \nabla z(x) + \kappa\nabla(j - z)(x)) - F(x, z(x), \nabla z(x))}{\kappa(j - z)(x)} (j - z)(x)dx.
\end{aligned} \tag{2.23}$$

Passing to the limit when κ tends to 0 and using the dominated convergence theorem, we obtain

$$\int_\Gamma \varphi(\nabla z(x))dx - \int_\Gamma \varphi(\nabla j(x))dx \geq \int_\Gamma \overline{f}(x, z(x), \nabla z(x))(j-z)(x)dx.$$

Then,

$$\int_\Gamma \varphi(\nabla z(x))dx + \int_\Gamma \overline{f}(x, z(x), \nabla z(x))z(x)dx \geq \int_\Gamma \overline{f}(x, z(x), \nabla z(x))j(x)dx + \int_\Gamma \varphi(\nabla j(x))dx,$$

for all $j \in D$. Consider any solution $w \in C^1(\Gamma)$ of (2.1). Let us set $v = f(., z(.), \nabla z(.))$ in (2.1). w and z maximize the functional ϕ_v over D. Then w and z are variational solutions of (2.1). Whence $u = z$. Then u is solution of the modified problem (2.14). As result u is solution of (1.1) and $u \in [\sigma, \rho]$. u maximizes Ψ over the set $\{s \in D : \sigma \leq s \leq \rho\}$. □

3. Main results

Theorem 3.1. *Suppose (H_f) and (H_Φ) hold. If the problem(1.1) admits a well ordred lower and upper solutions σ and ρ, then it has at least one solution u such that $\sigma \leq u \leq \rho$.*

Proof. This result follows immediatly from the proposition 2.9 □

Theorem 3.2. *Suppose (H_f) and (H_Φ) hold. If the problem(1.1) admits a lower solution σ, then it has at least one solution u such that $u \geq \sigma$.*

Proof. Let us consider the problem (2.17). Let us take $\lambda = \max\left\{\|\sigma\|_\infty, \frac{1}{2}\text{diam}\Gamma\right\}$ and $\overline{\rho} = \lambda + 1$. Then σ and $\overline{\rho}$ are a well ordered lower and upper solutions of (2.17). By proposition 2.9, (2.17) has at least one solution in $[\sigma, \frac{1}{2}\text{diam}\Gamma]$. It follows that (1.1) has at least one solution in $[\sigma, \frac{1}{2}\text{diam}\Gamma]$. □

Theorem 3.3. *Suppose (H_f) and (H_Φ) hold. If the problem(1.1) admits a upper solution ρ, then it has at least one solution u such that $u \leq \rho$.*

Proof. The proof is similaire to that of the theorem 3.2. □

Theorem 3.4. *Suppose (H_f) and (H_Φ) hold. If the problem(1.1) admits a strict lower solution σ and a strict upper solution, with $\sigma \nleq \rho$. Then it admits at least three solutions z_1, z_2, z_3 such that*

$$z_1 \leq z_2 \leq z_3, \quad z_1 \ll \rho, \quad z_1 \ngeq \sigma, \quad z_2 \nleq \rho, \quad z_3 \gg \sigma.$$

Proof. See the proof of proposition 2.10. □

Theorem 3.5. *Suppose (H_f) and (H_Φ) hold. If the problem(1.1) admits a lower solution σ and a upper solution ρ such that $\sigma \nleq \rho$, then it has at least two solutions z_1, z_2 such that*

$$z_1 < z_2, \quad z_1 \leq \rho \quad z_2 \geq \sigma.$$

Proof. Let σ and ρ be a not well ordered lower and upper solutions of (1.1). Let's set $\lambda = \max\left\{\|\sigma\|_\infty, \|\rho\|_\infty, \frac{1}{2}\text{diam }\Gamma\right\}$, $\overline{\sigma} = \lambda + 1$ and $\overline{\rho} = -\lambda - 1$. Then $\overline{\sigma}$ is strict lower-solution of (1.1) and $\overline{\rho}$ is a strict upper-solution of (1.1) such that $\overline{\sigma} \ll \min\{\sigma, \rho\}$ and $\overline{\rho} \gg \max\{\sigma, \rho\}$. Then, by proposition 2.9, there exist z_1, z_2 solutions of (1.1) such that $\overline{\sigma} < z_1 \leq \min\{\sigma, \rho\}$ and $\max\{\sigma, \rho\} \leq z_2 < \overline{\rho}$. So, $z_1 < z_2, \quad z_1 \leq \rho \quad z_2 \geq \sigma$. □

Theorem 3.6. *Suppose* (H_f) *and* (H_Φ) *hold. If the problem*(1.1) *admits a lower solution* σ, $\overline{\sigma}$ *and a upper solution* ρ, $\overline{\rho}$, *with* σ, ρ *strict,* $\overline{\sigma} \leq \min\{\sigma, \rho\} \leq \max\{\sigma, \rho\} \leq \overline{\rho}$ *and* $\sigma \nleq \rho$. *The problem admits at least Three solutions* z_1, z_2, z_3 *such that*

$$\overline{\sigma} \leq z_1 < z_2 < z_3 \leq \overline{\rho}, \quad z_2 \ngeq \sigma, \quad z_2 \nleq \rho, \quad z_3 \gg \sigma.$$

Proof. Let σ and ρ be a not well ordered strict lower and strict upper solutions of (1.1). Consider the functional intervals $[\overline{\sigma}, \min\{\sigma, \rho\}]$, $[\min\{\sigma, \rho\}, \max\{\sigma, \rho\}]$ and $[\max\{\sigma, \rho\}, \overline{\rho}]$. Since $\sigma \nleq \rho$ and σ, ρ strict, there exists a solution z_2 of (1.1) such that $\rho(x) < z_2(x) < \sigma(x)$, $\forall x \in \Gamma$. Then $z_2 \in]\min\{\sigma, \rho\}, \max\{\sigma, \rho\}[$. By proposition 2.9, each of intervals $[\overline{\sigma}, \min\{\sigma, \rho\}]$ and $[\max\{\sigma, \rho\}, \overline{\rho}]$ respectively contains solutions z_1, z_3 of (1.1). Then $\overline{\sigma}(x) \leq z_1(x) < z_2(x) < z_3(x) \leq \overline{\rho}(x)$, $\forall x \in \Gamma$. Therefore, we have

$$\overline{\sigma} \leq z_1 < z_2 < z_3 \leq \overline{\rho}, \quad z_2 \ngeq \sigma, \quad z_2 \nleq \rho, \quad z_3 \gg \sigma.$$

□

Theorem 3.7. *Suppose* (H_f) *and* (H_Φ) *hold. Suppose there exist a lower solution* σ *and an upper solution* ρ, *with* $\sigma \leq \rho$. *Then there is a solution* z *that maximizes the functional* Ψ *over the set* $\{z \in D : \sigma \leq z \leq \rho\}$, *where* D *is defined by* (2.2).

Proof. See the proof of proposition 2.11. □

Remark 3.8. Since $\|z\|_\infty < \dfrac{1}{2}\text{diam}\Gamma$, taking $-\max\left\{\|\rho\|_\infty, \frac{1}{2}\text{diam}\Gamma\right\}$ and $\max\left\{\|\sigma\|_\infty, \frac{1}{2}\text{diam}\Gamma\right\}$ as lower and upper solutions respectively , then by theorem 3.7, the problem (1.1) admits some solutions z_1 and z_2 that maximizes the functional Ψ over the sets $\{z \in D : z \leq \rho\}$ and $\{z \in D : \sigma \leq z\}$ respectively, where D is defined by (2.2).

4. Examples

A few application examples were given in [7], and we'll repeat them, in example 1, with a few slight modifications linked to the differential operator.

Example 1. *Let us consider the following problem:*

$$(4.1) \quad \begin{cases} -\ div \left(\dfrac{4 + \|\nabla z(x)\|^2 + \|\nabla z(x)\|^4}{(1 + \|\nabla z(x)\|^2)^2} \dfrac{\nabla z(x)}{\sqrt{1 - \|\nabla z(x)\|^2}} \right) = \lambda z^p(x) \quad in\ \Gamma \\ z = 0 \ on\ \partial\Gamma, \end{cases}$$

with $p > 0$ *and* $\lambda > 0$. *Here*

$$\Phi(u) = -\frac{4 + \|u\|^2 + \|u\|^4}{(1 + \|u\|^2)^2} \frac{u}{\sqrt{1 - \|u\|^2}}, \quad f(x, u, \nabla u) = \lambda u^p \quad \forall u \in \mathbb{R}.$$

In order to construct a strictly positive lower-solution σ to (4.1), consider the following problem

(4.2)
$$\begin{cases} -\ \text{div}\left(\dfrac{4+\|\nabla z(x)\|^2+\|\nabla z(x)\|^4}{(1+\|\nabla z(x)\|^2)^2}\dfrac{\nabla z(x)}{\sqrt{1-\|\nabla z(x)\|^2}}\right)=\lambda(z^+(x))^p \quad \text{in } \Omega \\ z=0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a fixed open ball of Γ and $\overline{\Omega}\subset\Gamma$, $z^+(x)=\max\{z(x),0\}$. From [6] proposition 2.7, the problem (4.2) admits at least one solution $z\in C^2(\overline{\Omega})$ and $z\gg 0$. Let us define the function $\sigma\in C^{0,1}(\overline{\Gamma})$ such that

$$\sigma(x)=\begin{cases} z^+(x) & \text{if } x\in\overline{\Omega} \\ 0 & \text{if } x\in\overline{\Gamma}\setminus\overline{\Omega} \end{cases}$$

then σ is a strict lower solution of (4.1)(for more details, see example 1 of [7]).

Remark 4.1. Examples 2,3,4 of [7] considered with operator Φ defined above, are also examples of applications.

Example 2.

(4.3)
$$\begin{cases} -\ \ div\left(\dfrac{4+2\|\nabla z(x)\|^2+3\|\nabla z(x)\|^4+\|\nabla z(x)\|^6}{(1+\|\nabla z(x)\|^2)\sqrt{1+(1+\|\nabla z(x)\|^2)^2}}\dfrac{\nabla z(x)}{\sqrt{1-\|\nabla z(x)\|^2}}\right)=-z(x) \\ +\lambda^*\|\nabla z(x)\|^q+\min\{1,\|x\|\} \quad in\ \Gamma \\ z=0\ on\ \partial\Gamma, \end{cases}$$

where $\lambda^*>0$, $q>0$ and

$$\Phi(u)=-\frac{4+2u^2+3u^4+u^6}{(1+u^2)\sqrt{1+(1+u^2)^2}}\sqrt{1-\|u\|^2} \quad \forall u\in\mathbb{R}^N.$$

The constant functions $\sigma(x)=0$ and $\rho(x)=2$ are respectively well-ordered lower and upper solutions of (4.3). Therefore, by theorems 3.1; 3.2; 3.3 and 3.7, there is at least one solution.

5. Conclusion

In this article, we have studied a second order nonlinear partial differential equation driven by the nonlinear differential operator $-$ div $(\Phi(\nabla u))$, with homogeneous Dirichlet boundary conditions. By combining the method of lower and upper solution, variational method and theory of topological degree, we have established existence of solution when the problem admits a unique lower solution or a unique upper solution and a multiplicity of solution when the problem admits a lower solution and upper solution which may or not well-ordered.

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